Approximate Fuzzy Reasoning

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Introduction

- What is „fuzzy logic" ?
- What is „modus ponens" ?
- How does one go from standard logic to fuzzy logic ?
- How do logic operands work and what is a t_norm, co_norm ?

Modus Ponens in Fuzzy Logic

In standard logic, Modus Ponens can be expressed as:

$$B' = A' \wedge (A \implies B)$$

In fuzzy logic, Modus Ponens can be expressed as:

$$B'(y) = \sup_{x \in X} \left(\mathcal{T} \left(A'(x), \mathcal{I}(A(x), B(y)) \right) \right)$$

Where:

- A is the first statement (antecedent)
- A' is the fact
- B is the second statement (consequent)
- B' is the conclusion

Now lets define a criteria (rule) $\mathcal{R}$ as:

$$\mathcal{R}(x, y) = \mathcal{I}(A(x), B(y))$$

In other words $\mathcal{R}$ can be expressed as:

$$\mathcal{R} : \text{IF } x \text{ is } A \text{ THEN } y \text{ is } B$$

Now lets expand our criteria to a set of criterias as: $\mathcal{R}_i$ as:

$$\mathcal{R}_i(x, y) = \mathcal{I}(A_i(x), B_i(y)), \; i = 1, 2..., m$$

Or expressed as:

$$\mathcal{R}_i = \text{IF } x_1 \text{ is } A_{i1} \text{ AND } x_2 \text{ is } A_{i2} \text{ AND } ... \; x_n \text{ is } A_{in} \text{ THEN } y \text{ is } B_i$$

$$A_{ij} \in \mathcal{F}(x_j) \text{ in rule } \mathcal{R}_i,$$

$$i = 1, 2..., m,$$

$$j = 1, 2..., n$$

FITA and FATI approach to Multicriteria Modus Ponens

- **FITA** - First Infer Then Aggregate

$$Z_{out} : B'(y) = \mathcal{T}\Big(B'_1(y), B'_2(y), ..., B'_m(y)\Big)$$

$$B'(y) = \mathcal{T}_{i=1}^{m}(B'_i(y)), \text{ know: } B'_i(y) = \mathcal{T}\Big(A'(x), \mathcal{R}_{\rangle}(x, y)\Big)$$

$$B'(y) = \mathcal{T}_{i=1}^{m}\left(\sup_{x \in X} \widehat{\mathcal{T}}(A'(x), \mathcal{R}_i(x, y))\right)$$

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- **FATI** - First Aggregate Then Infer

$$\widehat{\mathcal{R}}(x,y) = \mathcal{T}_{i=1}^{m}(\mathcal{R}_{\rangle}(x,y)) = \mathcal{T}_{i=1}^{m}\Big(\mathcal{I}(A_i(x), B_i(y))\Big)$$

$$\widehat{Z} : B'(y) = \widehat{\mathcal{T}}(A'(x), \widehat{\mathcal{R}}(x,y)) = \widehat{\mathcal{T}}\Big(A'(x), \mathcal{T}_{i=1}^{m}(\mathcal{R}_{\rangle}(x,y))\Big)$$

$$B'(y) = \sup_{x \in X} \widehat{\mathcal{T}}\Big(A'(x), \mathcal{T}_{i=1}^{m}\Big(\mathcal{I}(A_i(x), B_i(y))\Big)\Big)$$

- Prologue to **FATI** and **FITA**

$$\widehat{\mathcal{T}}\Big(A'(x), \Big(\bigcap R_i(x,y)\Big)\Big) \subseteq \bigcap\Big(\widehat{\mathcal{T}}\Big(A'(x), R_i(x,y)\Big)\Big)$$

SBR – Similarity Based Reasoning

- In SBR, we have to once again consider the fuzzy IF-THEN rule
- Let the given input be $\tilde{x}$ is A'

Inference in SBR schemes is based on:

- Measure of compatibility/similarity $M(A, A')$
- Modification function J

Matching Function M

- Any function $M\colon \mathcal{F}(X) \times \mathcal{F}(X)$ [1] $\Longrightarrow \langle 0, 1\rangle$ is said to be a matching function
- A matching function operates on $A,\ A'$ [2] to a give a real number in $\langle 0, 1\rangle$

Interpretation

- Degree of similarity between A and A'
- Subsethood measure of A' in A
- Measure of the compatibility of A' to A

[1] X – non-empty set, $\mathcal{F}(X)$ – fuzzy power set of X
[2] $A,\ A' \in \mathcal{F}(X)$

Matching Function Examples

- Zadeh's max-min:

$$M_Z(A, A') = \max_{x \in X} \min(A(x), A'(x))$$

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- Scalar product:

$$M_C(A, A') = \frac{A \cdot A'}{\max(A \cdot A, A' \cdot A')}$$

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- Disconsistency measure:

$$M_{TK}(A, A') = \sqrt{\frac{\sum_{i=1}^{n}(A(x_i) - A'(x_i))^2}{n}}$$

Modification Function J

- Let Y be a non-empty set and $B \in \mathcal{F}(Y)$
- The modification function J produces a modification $B' \in \mathcal{F}(Y)$ based on s[3] and B[4]

J is given by:

$$B'(y) = J(s, B(y)) = J(M(A, A'), B(y)), \quad y \in Y$$

[3] $s = M = (A, A')$
[4] B is the consequence in SBR

Modification Function Examples

- More or Less:

$$J_{\mathrm{ML}}(s, B) = B'(x) = \min\left(1, \frac{B(x)}{s}\right)$$

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$$J_{\mathrm{ML}}(s, B) = B'(x) = \min\left(1, \frac{B(x)}{s}\right)$$

- Membership Value Reduction:

$$J_{\mathrm{MVR}}(s, B) = B'(x) = B(x) \cdot s$$

- In the case of multiple rules in AR we infer the final output by aggregating over the rules
- We use an associative aggregation operation

$$G^5 : \langle 0, 1\rangle^2 \to \langle 0, 1\rangle$$

It is defined by the following:

$$B'(y) = G_{i=1}^{m}(J(M(A_i, A'), B_i(y))), \quad y \in Y$$

[5] G is either a t-norm or co-norm

Combiner Function K

- Let $K\colon \langle 0,1\rangle^2 \rightarrow \langle 0,1\rangle$, be an associative and commutative function that combines the matching degrees of A_j to A'_j for all $j = 1,\ldots,n$
- K is the combiner in the sequel

The consequence of an individual MISO rule is given by

$$B'(y) = J\Big(K\big(M(A_1, A'_1), \ldots, M(A_n, A'_n)\big), B(y)\Big),$$
$$= J\big(K(s_1, \ldots, s_n), B(y)\big), \quad y \in Y$$

We infer the final output by aggregating over the rules

$$B'(y) = G_{i=1}^{m}(J(K_{j=1}^{n}(M(A_{ij}, A'_j)), B(y))), \quad y \in Y$$

- Some SBR inference schemes along with their inference operations, where T is any t-norm, S is any t-conorm, I is any fuzzy implication and Avg. is the averaging operation

SBR scheme	G	J	K	M
CMI	T	I	T	M_Z
AARS	S_M	J_{MVR}, J_{ML}	Avg.	M_{TK}, M_Z
CDR	T_M	I	–	M_S

Properties of GMP 1/4

$$\begin{array}{llll} \textit{if} & x \textit{ is } A \textit{ then} & y \textit{ is } B \\ & x \textit{ is } A & \\ \hline & & y \textit{ is } B \end{array}$$

Figure: **Basic property** of GMP (1)

- Rule : if x is A then y is B
- Fact : x is A
- Conclusion : y is B

#basic property

$$B'(y) = \sup_{x \in X'} \mathcal{T}\Big(A(x), \mathcal{I}\big(A(x), B(y)\big)\Big), y \in Y$$

- Analyzed by when generalized modus ponens coincides with classical modus ponens

Properties of GMP 2/4

$$
\begin{array}{ll}
\textit{if } x \textit{ is } A \textit{ then} & y \textit{ is } B \\
x \textit{ is } A' \subset A & \\
\hline
 & y \textit{ is } B
\end{array}
$$

Figure: **Subset property** of GMP (1)

Fuzzy subset

$$A \subset B \Longleftrightarrow ((\forall x \in X : \mu_A(x) \leq \mu_B(x)) \wedge A \neq B)$$

#subset property

Proof

$$B''(y) = A'' \circ \mathcal{R}(A, B)$$

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$$B''(y) = \sup_{x \in X} \mathcal{T}\Big(A''(x), \mathcal{I}\big(A(x), B(y)\big)\Big)$$

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$$B''(y) = \sup_{x \in X} \mathcal{T}\Big(A''(x), \mathcal{I}\big(A(x), B(y)\big)\Big)$$

$$B''(y) \geq \sup_{x \in X} \mathcal{T}\Big(A'(x), \mathcal{I}\big(A(x), B(y)\big)\Big)$$

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$$B''(y) \geq \sup_{x \in X} \mathcal{T}\Big(A'(x), \mathcal{I}\big(A(x), B(y)\big)\Big)$$

$$B''(y) = B'(y)$$

Properties of GMP 3/4

$$\begin{array}{ll} \textit{if } x \textit{ is } A \textit{ then} & y \textit{ is } B \\ x \textit{ is } \neg A & \\ \hline & y \textit{ is unknown} \end{array}$$

Figure: **Total indeterminance** of GMP (1)

$\neg A$ represents negation of fuzzy set A by some fuzzy negator

#total indeterminance

$$B'(y) = \sup_{x \in X} \mathcal{T}\Big(\overline{A}(x), \mathcal{I}\big(A(x), B(y)\big)\Big)$$

[6] $X' = \{x \in X \mid \overline{A}(x) = 1\}$

$$B'(y) = \sup_{x \in X} \mathcal{T}\Big(\overline{A}(x), \mathcal{I}\big(A(x), B(y)\big)\Big)$$

$$B'(y) \geq \sup_{x \in X'} \mathcal{T}\Big(\overline{A}(x), \mathcal{I}\big(A(x), B(y)\big)\Big)^{6}$$

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$$B'(y) = \sup_{x \in X'} \mathcal{T}\Big(\overline{A}(x), \mathcal{I}\big(0, B(y)\big)\Big)$$

$$B'(y) = \mathcal{T}(1, 1) = 1$$

[6] $X' = \{x \in X \mid \overline{A}(x) = 1\}$

Properties of GMP 4/4

$$\begin{array}{lll} \textit{if} & x \textit{ is } A \textit{ then} & y \textit{ is } B \\ & x \textit{ is } A' & \\ \hline & & y \textit{ is } B' \supset B \end{array}$$

Figure: **Superset property** of GMP (1)

Fuzzy superset

$$A \supset B \iff ((\forall x \in X : \mu_A(x) \geq \mu_B(x)) \wedge A \neq B)$$

#superset property

Proof

We have to consider 2 cases:

- Fuzzy set A′ is a fuzzy singleton
- set A′ is not a singleton

Consider matching function $\mathcal{M}$ and modification function $\mathcal{J} = \mathcal{I}$, a fuzzy implication (2)

- Superset property

$$\mathcal{I}(p, q) \geq q, \quad p, q \in [0, 1]$$

- Subset property

$$A_1 \subseteq A_2 \Longrightarrow \mathcal{M}(A_1, A) \geq M(A_2, A)$$

- Total indeterminance

$$\mathcal{M}(A, \overline{A}) = \mathcal{M}(\overline{A}, A) = 0$$

- Basic property

$$\mathcal{M}(A, A) = 1 \;\wedge\; \mathcal{I}(1, y) = y$$

References

- Christer, C.; Fuller, R.: Fuzzy Reasoning in Decision Making and Optimization. Physica-Verlag HD, 2012, ISBN 9783790818055, 338 s.
- Michał Baczyński, B. J.: Fuzzy Implications. Springer, 2008, ISBN 9783540690825, 310 s.

Python notebook - visualization :

https://colab.research.google.com/drive/1ILogFsFL94pFxomR8eR_qon226fNSoS4?usp=sharing