Introduction to Fuzzy Optimization

Mathematical Foundations of Fuzzy Logic

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What is Fuzzy Optimization?

Definition:
Fuzzy optimization extends traditional optimization methods to handle uncertainty and imprecision by incorporating fuzzy logic. It allows decision-making in systems where:

- Constraints and objectives are not precisely defined.
- Parameters are expressed using fuzzy numbers or linguistic terms.
- Solutions are evaluated based on degrees of satisfaction rather than strict feasibility.

Key Feature:
Fuzzy optimization provides a framework to find solutions that best satisfy all fuzzy constraints and objectives simultaneously.

Applications of Fuzzy Optimization

Where can Fuzzy Optimization be used?

- **Decision-Making Problems:** Handling multi-criteria decision-making under uncertainty.
- **Supply Chain Management:** Optimizing resource allocation and logistics with imprecise demand or supply forecasts.
- **Control Systems:** Designing robust controllers in dynamic environments with uncertain parameters.
- **Finance and Economics:** Portfolio optimization, risk analysis, and economic modeling with vague or uncertain data.
- **Engineering Design:** Solving optimization problems with flexible constraints and objectives.

Key Insight:

Fuzzy optimization is widely applicable in systems requiring flexibility and adaptability to uncertainty.

1. Possibilistic linear equality systems

Linear equality system

- Modelling real world problems mathematically we often have to find a solution to a **linear equality system**.

$$a_{i1}x_1 + \cdots + a_{in}x_n = b_i, \quad i = 1, \ldots, m$$

or shortly,

$$Ax = b$$

where a_{ij}, b_i, and x_j are **real numbers**.
- Linear equality system generally belongs to the class of **ill-posed problems**.
- Small perturbation of the parameters a_{ij} and b_i may cause a large deviation in the solution.

Possibilistic linear equality system

- A **possibilistic linear equality system** is

$$\tilde{a}_{i1}x_1 + \cdots + \tilde{a}_{in}x_n = \tilde{b}_i, \quad i = 1, \ldots, m$$

  or shortly,

$$\tilde{A}x = \tilde{b}$$

  where $\tilde{a}_{ij}, \tilde{b}_i \in F(\mathbb{R})$ are **fuzzy quantities**, $x \in \mathbb{R}^n$
- The operations addition and multiplication by a real number of fuzzy quantities are defined by Zadeh's extension principle.
- The equation is understood in possibilistic sense.

Degree of satisfaction

- Recall the **truth value** of the assertion "$\tilde{a}$ is equal to $\tilde{b}$"

$$\mathrm{Pos}(\tilde{a} = \tilde{b}) = \sup_t \{\tilde{a}(t) \wedge \tilde{b}(t)\} = (\tilde{a} - \tilde{b})(0)$$

- We denote by $\mu_i(x)$ the **degree of satisfaction** of the i-th equation in the possibilistic linear equality system at the point $x \in \mathbb{R}^n$

$$\mu_i(x) = \mathrm{Pos}(\tilde{a}_{i1}x_1 + \cdots + \tilde{a}_{in}x_n = \tilde{b}_i)$$

- The **fuzzy solution** of the possibilistic linear equality system can be viewed as the intersection of the μ_i's such that

$$\mu(x) = \min\{\mu_1(x), \ldots, \mu_m(x)\}$$

Maximizing solution

- A **measure of consistency** for the possibilistic linear equality system is defined as

$$\mu^* = \sup\{\mu(x) \mid x \in \mathbb{R}^n\}$$

- Let X^* be the set of points $x \in \mathbb{R}^n$ for which $\mu(x)$ attains its **maximum**

$$X^* = \{x^* \in \mathbb{R}^n \mid \mu(x^*) = \mu^*\}$$

- If $X^* \neq \emptyset$ and $x^* \in X^*$, then x^* is called a **maximizing** (or best) **solution** of the possibilistic linear equality system.

Hausdorff distance

- If $\tilde{a}$ and $\tilde{b}$ are fuzzy numbers with

$$[a]^\alpha = [a_1(\alpha), a_2(\alpha)] \quad \text{and} \quad [b]^\alpha = [b_1(\alpha), b_2(\alpha)]$$

  also known as α-level or α-cut of a fuzzy number.
- Then their **Hausdorff distance** is defined as

$$D(\tilde{a}, \tilde{b}) = \sup_{\alpha \in [0,1]} \max\{|a_1(\alpha) - b_1(\alpha)|, |a_2(\alpha) - b_2(\alpha)|\}$$

- $D(\tilde{a}, \tilde{b})$ is the **maximal distance** between the α-level sets of $\tilde{a}$ and $\tilde{b}$

Lipschitz condition

- Let $L > 0$ be a **real number**.
- By $F(L)$ we denote the set of all fuzzy numbers $\tilde{a} \in F$ with membership function satisfying the **Lipschitz condition**

$$|\tilde{a}(t) - \tilde{a}(t')| \leq L|t - t'|, \quad \forall t, t' \in \mathbb{R}$$

- The **Lipschitz condition** is a criterion that ensures a function does not change too rapidly.
- In fuzzy mathematics, membership functions satisfying the **Lipschitz condition** ensure that the fuzziness changes smoothly with respect to its parameters.

Perturbed fuzzy equality system, part I

- In many important cases, the fuzzy parameters $\tilde{a}_{ij}, \tilde{b}_i$ of the possibilistic linear equality system are **not known exactly**.
- We have to work with their **approximations** $\tilde{a}_{ij}^{\delta}, \tilde{b}_i^{\delta}$ such that

$$\max_{i,j} D(\tilde{a}_{ij}, \tilde{a}_{ij}^{\delta}) \leq \delta \quad \max_{i} D(\tilde{b}_i, \tilde{b}_i^{\delta}) \leq \delta$$

  where $\delta \geq 0$ is a **real number**.
- Then we get the following system with **perturbed fuzzy parameters**

$$\tilde{a}_{i1}^{\delta} x_1 + \cdots + \tilde{a}_{in}^{\delta} x_n = \tilde{b}_i^{\delta}, \quad i = 1, \ldots, m$$

  or shortly,

$$\tilde{A}^{\delta} x = \tilde{b}^{\delta}$$

Perturbed fuzzy equality system, part II

- In a similar manner, we define the **degree of satisfaction** of the i-th equation of the perturbed fuzzy equality system at $x \in \mathbb{R}^n$

$$\mu_i^\delta(x) = \mathrm{Pos}(\tilde{a}_{i1}^\delta x_1 + \cdots + \tilde{a}_{in}^\delta x_n = \tilde{b}_i^\delta)$$

- In a similar manner, we define the **fuzzy solution** of the perturbed fuzzy equality system

$$\mu^\delta(x) = \min\{\mu_1^\delta(x), \ldots, \mu_m^\delta(x)\}$$

- In a similar manner, we define the **measure of consistency** of the perturbed fuzzy equality system

$$\mu^*(\delta) = \sup\{\mu^\delta(x) \mid x \in \mathbb{R}^n\}$$

- Let $X^*(\delta)$ denote the set of **maximizing solutions** of the perturbed fuzzy equality system.

1st theorem on stability of fuzzy solutions

- Let $L > 0$ and $\tilde{a}_{ij}, \tilde{a}_{ij}^{\delta}, \tilde{b}_i, \tilde{b}_i^{\delta} \in F(L)$
- If the **Hausdorff distance** between fuzzy numbers and perturbed fuzzy numbers is bounded by δ:

$$\max_{i,j} D(\tilde{a}_{ij}, \tilde{a}_{ij}^{\delta}) \leq \delta, \quad \max_{i} D(\tilde{b}_i, \tilde{b}_i^{\delta}) \leq \delta$$

- Then the fuzzy solutions $\mu(x)$ and $\mu^{\delta}(x)$ of fuzzy system and perturbed fuzzy system, respectively, satisfy:

$$\|\mu - \mu^{\delta}\|_{\infty} = \sup_{x \in \mathbb{R}^n} |\mu(x) - \mu^{\delta}(x)| \leq L\delta$$

- **Key Idea:** The difference between the solutions is bounded by $L\delta$, ensuring stability under small perturbations of the fuzzy parameters.
- **Extension:** This theorem applies to possibilistic linear equality systems with continuous fuzzy numbers.

Fuzzy solution (symmetric triangular fuzzy numbers)

- Consider now the possibilistic equality system with **symmetric triangular fuzzy numbers**

$$(a_{i1}, \alpha)x_1 + \cdots + (a_{in}, \alpha)x_n = (b_i, \alpha), \quad i = 1, \ldots, m$$

or shortly,

$$(A, \alpha)x = (b, \alpha)$$

- Then the **fuzzy solution** can be written as

$$\mu(x) = \begin{cases} 1 & \text{if } Ax = b \\ 1 - \dfrac{||Ax - b||_\infty}{\alpha(|x|_1 + 1)} & \text{if } 0 < ||Ax - b||_\infty \leq \alpha(|x|_1 + 1) \\ 0 & \text{if } ||Ax - b||_\infty > \alpha(|x|_1 + 1) \end{cases}$$

- Where

$$\|Ax - b\|_\infty = \max\{|(a_1, x) - b_1|, \ldots, |(a_m, x) - b_m|\}$$

2nd theorem on stability of fuzzy solutions

- If

$$D(\tilde{A}, \tilde{A}^\delta) = \max_{i,j} |a_{ij} - a_{ij}^\delta| \leq \delta, \quad D(\tilde{b}, \tilde{b}^\delta) = \max_{i} |b_i - b_i^\delta| \leq \delta$$

- Then

$$\|\mu - \mu^\delta\|_\infty = \sup_x |\mu(x) - \mu^\delta(x)| \leq \frac{\delta}{\alpha}$$

- Where $\mu(x)$ and $\mu^\delta(x)$ are the fuzzy solutions of the systems

$$(A, \alpha)x = (b, \alpha) \quad \text{and} \quad (A^\delta, \alpha)x = (b^\delta, \alpha)$$

  respectively.
- **Key Idea:** The difference between the solutions is bounded by $\frac{\delta}{\alpha}$, ensuring stability under small perturbations of the fuzzy parameters.

3rd theorem on stability of fuzzy solutions

- Let $\tilde{a}_{ij}, \tilde{a}_{ij}^{\delta}, \tilde{b}_i, \tilde{b}_i^{\delta} \in F$ be fuzzy numbers.
- If the **Hausdorff distance** between fuzzy numbers and perturbed fuzzy numbers is bounded by δ:

$$\max_{i,j} D(\tilde{a}_{ij}, \tilde{a}_{ij}^{\delta}) \leq \delta, \quad \max_{i} D(\tilde{b}_i, \tilde{b}_i^{\delta}) \leq \delta$$

- Then

$$\|\mu - \mu^{\delta}\|_{\infty} \leq \omega(\delta),$$

- **Key Idea:** The difference between the solutions is bounded by $\omega(\delta)$, ensuring stability under small perturbations of the fuzzy parameters.
- The **modulus of continuity** $\omega(\delta)$ is a function that measures how sensitive the fuzzy coefficients are to small changes or perturbations.
- It represents the maximum amount by which the fuzzy solutions $\mu(x)$ and $\mu^{\delta}(x)$ can differ due to a perturbation δ in the fuzzy parameters.

Example of fuzzy solution, part I

- Consider the following two-dimensional possibilistic equality system

$$(1, \alpha)x_1 + (1, \alpha)x_2 = (0, \alpha)$$

$$(1, \alpha)x_1 - (1, \alpha)x_2 = (0, \alpha)$$

- Then its **fuzzy solution** is

$$\mu(x) = \begin{cases} 1 & \text{if } x = 0 \\ \tau_2(x) & \text{if } 0 < \max\{|x_1 - x_2|, |x_1 + x_2|\} \leq \alpha(|x_1| + |x_2| + 1) \\ 0 & \text{if } \max\{|x_1 - x_2|, |x_1 + x_2|\} > \alpha(|x_1| + |x_2| + 1) \end{cases}$$

- Where

$$\tau_2(x) = 1 - \frac{\max\{|x_1 - x_2|, |x_1 + x_2|\}}{\alpha(|x_1| + |x_2| + 1)}$$

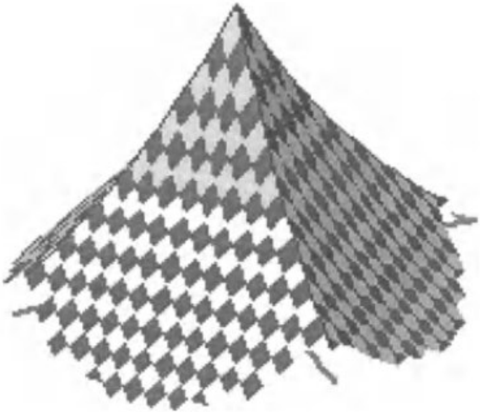

Fig. 4.1. The graph of fuzzy solution of system with $\alpha = 0.4$

Example of fuzzy solution, part III

- The only **maximizing solution** of system is

$$x^* = (0, 0)$$

- There is no problem with the stability of the solution even for the **crisp system**

$$\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

- Because $\det(A) \neq 0$

2. Sensitivity analysis of $\tilde{a}x = \tilde{b}$ and $\tilde{a}^{\delta}x = \tilde{b}^{\delta}$

Crisp vs Fuzzy

- **Crisp Systems:** Use sharp, binary logic (0 or 1).
- **Fuzzy Systems:** Allow for degrees of membership between 0 and 1.
- **Key Question:** How do these approaches relate, and what are their differences?

Crisp vs Fuzzy Systems: A Comparison

Crisp Systems:

- Sharp boundaries.
- Boolean logic (true/false).
- Example: Is age ≥ 18?

Fuzzy Systems:

- Smooth transitions.
- Degrees of membership ($0 \leq \mu \leq 1$).
- Example: "Almost an adult" ($\mu = 0.7$).

Example: Automation Control

Crisp:

- Rule: If temperature $\geq 30^\circ C$, turn on the fan.
- $t = 29^\circ C$: Fan off.

Fuzzy:

- Rule: If temperature is close to $30^\circ C$, the fan runs partially.
- $t = 29^\circ C$: Fan runs at 80
- $t = 31^\circ C$: Fan runs at 100

Relationship Between Crisp and Fuzzy

- Crisp systems are a special case of fuzzy systems with strict membership functions ($\mu = 0$ or $\mu = 1$).
- Fuzzy systems generalize crisp systems by introducing smooth transitions.

Advantages of Fuzzy Systems

- Flexibility: Models real-world uncertainty.
- Smoothness: Eliminates rigid boundaries.
- Real-world applications:
    - Artificial Intelligence.
    - Control Systems.
    - Decision-making in uncertain environments.

Representations of Triangular Fuzzy Numbers

Fuzzy numbers can be represented in two ways:

1. **Representation as a triple:** (a_1, a_2, a_3)
   - a_1: left boundary ($\mu(x) = 0$);
   - a_2: central value ($\mu(x) = 1$);
   - a_3: right boundary ($\mu(x) = 0$).

Example:

$$\tilde{a} = (1, 2, 3)$$

2. **Representation as a pair:** (a, α)
   - a: central value ($\mu(x) = 1$);
   - α: distance from the center to the boundaries ($\alpha = a_2 - a_1$).

Example:

$$\tilde{a} = (2, 1)$$

Equivalent representation:

$$\tilde{a} = (1, 2, 3)$$

Problem Statement

We analyze the sensitivity of the fuzzy system:

$$\tilde{a}x = \tilde{b},$$

where:

- $\tilde{a} = (a, \alpha)$ and $\tilde{b} = (b, \alpha)$,
- These are fuzzy numbers of symmetric triangular form with width $\alpha > 0$,
- The system represents a possibilistic equality.

Goal: To find the fuzzy solution $\mu(x)$.

Maximizing Solution

For $a \neq 0$, the unique maximizing solution is:

$$x^* = \frac{b}{a},$$

which corresponds to the crisp solution of $ax = b$.

Key Observation:

- This solution x^* provides the highest possibility $\mu(x) = 1$.
- For all other values of x, $\mu(x) \leq 1$.

Consider a perturbed system:

$$\tilde{a}^{\delta} x = \tilde{b}^{\delta},$$

where:

$$|\tilde{a} - a| \leq \delta, \quad |\tilde{b} - b| \leq \delta.$$

Key Result:

- The difference between the fuzzy solutions of the original and perturbed systems is bounded:

$$\|\mu(x) - \mu^{\delta}(x)\| \leq \frac{\delta}{\alpha}.$$

Interpretation: Small perturbations δ result in proportional changes to the fuzzy solution.

Example 1

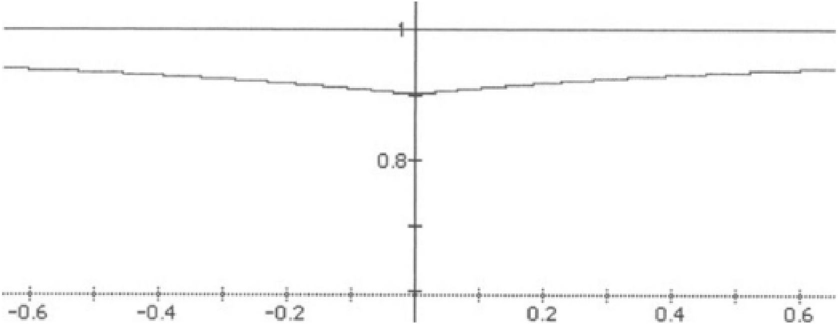

Fig. 4.4. Fuzzy solution of $(0,\alpha)x = (b^\delta, \alpha)$ with $\alpha = 0.2$ and $\delta = 0.02$.

Example 2

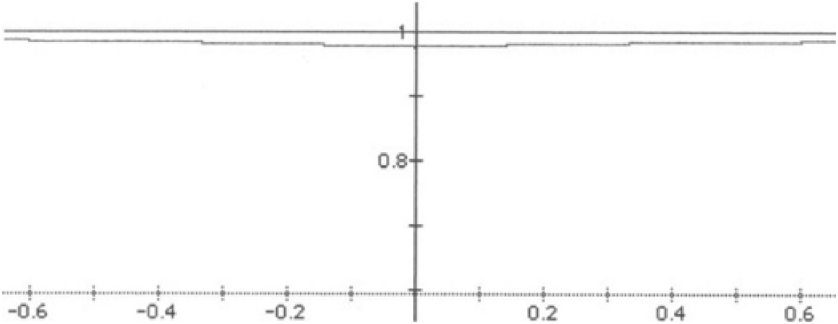

Fig. 4.5. Fuzzy solution of $(0, \alpha)x = (b^\delta, \alpha)$ with $\alpha = 0.2$ and $\delta = 0.005$.

Example 3

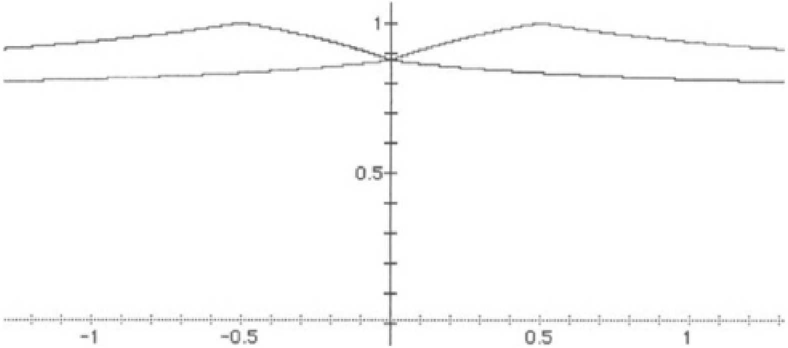

Fig. 4.7. Fuzzy solutions of $(a^\delta, \alpha)x = (b^\delta, \alpha)$ and $(a, \alpha)x = (b, \alpha)$ with $a^\delta = -0.01$, $a = 0.01$, $b^\delta = b = 0.005$, $\alpha = 0.04$ and $\delta = 0.02$. The maximizing solutions are $x^*(\delta) = -0.5$ and $x^* = 0.5$.

3. Possibilistic systems with trapezoid fuzzy numbers

Fuzzy equality system (trapezoid fuzzy numbers)

- Consider now a possibilistic linear equality system:

$$\tilde{a}_{i1}x_1 + \cdots + \tilde{a}_{in}x_n = \tilde{b}_i, \quad i = 1, \dots, m$$

- Where $\tilde{a}_{ij} \in F$ and $\tilde{b}_i \in F$ are **symmetric trapezoid fuzzy numbers** with the same width $\alpha > 0$ and tolerance intervals $[a_{ij} - \theta, a_{ij} + \theta]$ and $[b_i - \theta, b_i + \theta]$, respectively.
- Represented by $A = (a, b, \alpha, \beta)$ as:

$$\tilde{a}_{ij} = (a_{ij} - \theta, a_{ij} + \theta, \alpha, \alpha)$$

and

$$\tilde{b}_i = (b_i - \theta, b_i + \theta, \alpha, \alpha)$$

Symmetric trapezoid fuzzy number

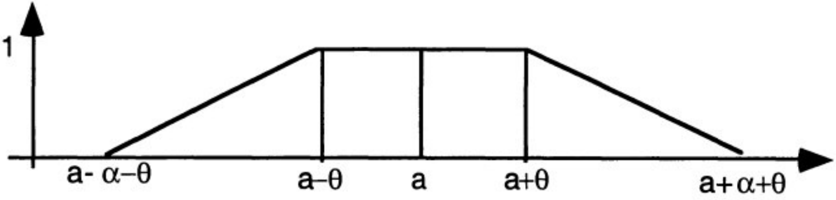

Fig. 4.8. A symmetric trapezoid fuzzy number with center a

Perturbed fuzzy equality system (trapezoid fuzzy numbers)

- The perturbed possibilistic linear equality system with **symmetric trapezoid fuzzy numbers** is

$$\tilde{a}^{\delta}_{i1}x_1 + \cdots + \tilde{a}^{\delta}_{in}x_n = \tilde{b}^{\delta}_i, \quad i = 1, \ldots, m$$

- Where

$$\tilde{a}^{\delta}_{ij} = (a^{\delta}_{ij} - \theta, a^{\delta}_{ij} + \theta, \alpha, \alpha)$$

and

$$\tilde{b}^{\delta}_i = (b^{\delta}_i - \theta, b^{\delta}_i + \theta, \alpha, \alpha)$$

Fuzzy solutions (symmetric trapezoid fuzzy numbers)

- The **fuzzy solutions** to trapezoid fuzzy system and perturbed trapezoid fuzzy system can be written as

$$\mu(x) = \begin{cases} 1 & \text{if } ||Ax - b||_\infty \le \theta(|x|_1 + 1) \\ 1 + \dfrac{\theta}{\alpha} - \dfrac{||Ax - b||_\infty}{\alpha(|x|_1 + 1)} & \text{if } \theta(|x|_1 + 1) < ||Ax - b||_\infty \le (\theta + \alpha)(|x|_1 + 1) \\ 0 & \text{if } ||Ax - b||_\infty > (\theta + \alpha)(|x|_1 + 1) \end{cases}$$

and

$$\mu^\delta(x) = \begin{cases} 1 & \text{if } ||A^\delta x - b^\delta||_\infty \le \theta(|x|_1 + 1) \\ 1 + \dfrac{\theta}{\alpha} - \dfrac{||A^\delta x - b^\delta||_\infty}{\alpha(|x|_1 + 1)} & \text{if } \theta(|x|_1 + 1) < ||A^\delta x - b^\delta||_\infty \le (\theta + \alpha)(|x|_1 + 1) \\ 0 & \text{if } ||A^\delta x - b^\delta||_\infty > (\theta + \alpha)(|x|_1 + 1) \end{cases}$$

- The stability property of fuzzy solutions does not depend on θ

4th theorem on stability of fuzzy solutions (trapezoid case)

- Let $\delta > 0$ and let μ and μ^δ be the solutions of possibilistic equality systems with **symmetric trapezoid fuzzy numbers**.
- If $a_{ij}, a_{ij}^\delta, b_i$ and b_i^δ satisfy:

$$D(\tilde{a}_{ij}, \tilde{a}_{ij}^\delta) = |a_{ij} - a_{ij}^\delta| \leq \delta, \quad D(\tilde{b}_i, \tilde{b}_i^\delta) = |b_i - b_i^\delta| \leq \delta$$

- Then

$$\|\mu - \mu^\delta\|_\infty = \sup_{x \in \mathbb{R}^n} |\mu(x) - \mu^\delta(x)| \leq \frac{\delta}{\alpha}$$

- **Key Idea:** The difference between the solutions is bounded by $\frac{\delta}{\alpha}$, ensuring stability under small perturbations of the fuzzy parameters.

Example of fuzzy solution (trapezoid case)

- The **fuzzy solution** of a possibilistic equation

$$(a-\theta, a+\theta, \alpha, \alpha)x = (b-\theta, b+\theta, \alpha, \alpha)$$

  can be written as

$$\mu(x) = \begin{cases} 1 & \text{if } |ax-b| \leq \theta(|x|+1) \\ 1 + \dfrac{\theta}{\alpha} - \dfrac{|ax-b|}{\alpha(|x|+1)} & \text{if } \theta(|x|+1) < |ax-b| \leq (\theta+\alpha)(|x|+1) \\ 0 & \text{if } |ax-b| > (\theta+\alpha)(|x|+1) \end{cases}$$

- It is clear that the **set of maximizing solutions**

$$X^* = \{x \in \mathbb{R} : |ax-b| \leq \theta(|x|+1)\}$$

  always contains the (crisp) solution set, X^{**}, of the equality $ax = b$

4. Flexible linear programming

Classical LP model:

$$\langle a_0, x \rangle \to \min \quad \text{subject to } Ax \leq b$$

Alternative formulation:

$$a_{01}x_1 + \cdots + a_{0n}x_n \leq b_0,$$

where b_0 is a predefined aspiration level.

Flexible Linear Programming (FLP)

Replacement of crisp parameters:

- $a_{ij} \to \tilde{a}_{ij} = (a_{ij}, \alpha)$
- $b_i \to \tilde{b}_i = (b_i, d_i)$,

where $\alpha > 0, d_i > 0$ represent tolerance levels.

Degree of satisfaction

$$\mu_i(x) = \begin{cases} 1, & \text{if } \langle a_i, x \rangle \leq b_i, \\ 1 - \frac{(a_i,x) - b_i}{\alpha |x| + d_i}, & \text{otherwise.} \\ 0, & \text{if } \langle a_i, x \rangle > b_i + \alpha |x|_1 + d_i. \end{cases}$$

When $\alpha = 0$:

$$\mu_i(x) = \begin{cases} 1, & \text{if } \langle a_i, x \rangle \leq b_i, \\ 1 - \frac{\langle a_i,x \rangle - b_i}{d_i}, & \text{if } b_i < \langle a_i, x \rangle \leq b_i + d_i, \\ 0, & \text{if } \langle a_i, x \rangle > b_i + d_i. \end{cases}$$

Interpretation of $\mu_i(x)$

- If $\mu_i(x) = 1$, then x fully satisfies the constraint.
- If $0 < \mu_i(x) < 1$, then x partially satisfies the constraint.
- If $\mu_i(x) = 0$, the violation of the constraint is unacceptable.

Perturbed Fuzzy Linear Programming (FLP) Problem

Problem Statement

The perturbed FLP problem (4.27) is given by:

$$(a_{i1}^{\delta}\alpha)x_1 + \cdots + (a_{in}^{\delta}\alpha)x_n \leq (b_i^{\delta}, d_i), \quad i = 0, \ldots, m,$$

where the coefficients satisfy (4.28):

$$\max_{i,j} |a_{ij} - a_{ij}^{\delta}| \leq \delta, \quad \max_i |b_i - b_i^{\delta}| \leq \delta.$$

Theorem: Flexible Linear Programming (FLP)

Statement:
Let $\mu(x)$ and $\mu^\delta(x)$ be solutions of FLP problems (4.24) and (4.27), respectively. Then:

$$\|\mu - \mu^\delta\|_\infty = \sup_{x \in \mathbb{R}^n} |\mu(x) - \mu^\delta(x)| \leq \delta \left[\frac{1}{\alpha} + \frac{1}{d}\right]$$

where:

$$d = \min\{d_0, d_1, \ldots, d_m\}.$$

Key Implications:

- Provides a bound on the difference between the solutions of FLP problems under parameter δ.
- Involves constants α and d derived from problem parameters.

Overview of the Proof

- Goal: Prove that the absolute difference $|\mu_i(x) - \mu_i^\delta(x)|$ is bounded for all cases.
- Approach: Analyze all possible combinations of $\mu_i(x)$ and $\mu_i^\delta(x)$.

Case 1: $\mu_i(x) = \mu_i^\delta(x)$

- If $\mu_i(x) = \mu_i^\delta(x)$, then:

$$|\mu_i(x) - \mu_i^\delta(x)| = 0$$

- This is trivial, and Equation (4.31) is satisfied.

Case 2: $0 < \mu_i(x) < 1$ and $0 < \mu_i^\delta(x) < 1$

- Analyze the difference:

$$|\mu_i(x) - \mu_i^\delta(x)| = \left|1 - \frac{(a_i, x) - b_i}{\alpha|x|_1 + d_i} - \left(1 - \frac{(a_i^\delta, x) - b_i^\delta}{\alpha|x|_1 + d_i}\right)\right|$$

- Using the bounds on (a_i, x) and constants α, d_i, derive:

$$|\mu_i(x) - \mu_i^\delta(x)| \leq \delta \cdot \left[\frac{1}{\alpha} + \frac{1}{d_i}\right]$$

Case 3: $\mu_i(x) = 1$, $0 < \mu_i^\delta(x) < 1$

- For $\mu_i(x) = 1$, we have $(a_i, x) \le b_i$.
- Substituting, the difference becomes:

$$|\mu_i(x) - \mu_i^\delta(x)| = \left|1 - \left[1 - \frac{(a_i^\delta, x) - b_i^\delta}{\alpha|x|_1 + d_i}\right]\right|$$

Case 4: $0 < \mu_i(x) < 1$, $\mu_i^\delta(x) = 1$

- This case mirrors Case 3 with roles of $\mu_i(x)$ and $\mu_i^\delta(x)$ reversed.
- The derived bound remains:

$$|\mu_i(x) - \mu_i^\delta(x)| \leq \delta \cdot \left[\frac{1}{\alpha} + \frac{1}{d_i}\right]$$

Case 5: $0 < \mu_i(x) < 1$, $\mu_i^\delta(x) = 0$

- For $\mu_i^\delta(x) = 0$, we have:

$$(a_i^\delta, x) - b_i^\delta > \alpha|x|_1 + d_i$$

- The absolute difference becomes:

$$|\mu_i(x) - \mu_i^\delta(x)| = \left|1 - \frac{(a_i, x) - b_i}{\alpha|x|_1 + d_i}\right|$$

- This is shown to satisfy the bounded condition.

Case 6: $\mu_i(x) = 0,\ 0 < \mu'_i(x) < 1$

- This case is symmetric to Case 5.
- The bound remains valid.

Case 7: $\mu_i(x) = 1$, $\mu_i^\delta(x) = 0$

- Assume $\mu_i(x) = 1$, $\mu_i^\delta(x) = 0$. From (4.28), this leads to:

$$|(a_i, x) - b_i - ((a_i^\delta, x) - b_i^\delta)| \leq \delta(|x|_1 + 1),$$

- On the other hand we have:

$$|(a_i, x) - b_i - ((a_i^\delta, x) - b_i^\delta)| > \delta(|x|_1 + 1),$$

- Thus, this case is not feasible.

That's all Folks!